M.Phil./Ph.D. (Mathematics) Entrance Examination 2016-17

Max Time: 2 hours Max Marks: 150

Instructions: There are 50 questions. Every question has four choices of which exactly one is correct. *For correct answer, 3 marks will be given. For wrong answer, 1 mark will be deducted.* Scientific calculators are allowed.

In the following $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of all real numbers, natural numbers, rational numbers, integers and complex numbers respectively.

(1) We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$. The series $\displaystyle\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n$ converges to

A) 1. B) 2. C) 0. D) $\frac{1}{2}$.

(2) Which of the following statements is correct?

A) Intersection of two connected sets in a topological space is connected.
B) Countable union of compact sets in a topological space is compact.
C) Closure of a compact set in a topological space is compact.
D) None of A), B), C) is correct.

(3) An uncountable set with co-countable topology is

A) second countable.
B) first countable but not second countable.
C) not first countable.
D) separable.

(4) Which of the following statements is true?

A) A convergent net in a topological space has a unique limit.
B) Only eventually constant nets are convergent in an infinite space with cofinite topology.
C) If X and Y are topological spaces and $f : X \to Y$ be a function. f is continuous on X if and only if $f(\lim_n x_n) = \lim_n f(x_n)$ for every convergent sequence (x_n) in X.
D) None of A), B), C) is true.

(5) Let τ be the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and the sets $(a, b) \sim A$, where $A = \{1/n : n \in \mathbb{N}\}$. Then

A) τ is not comparable with lower limit topology on $\mathbb{R}$.
B) τ is strictly coarser than the usual topology on $\mathbb{R}$.
C) τ is not comparable with co-finite topology on $\mathbb{R}$.
D) τ is strictly finer than upper limit topology on $\mathbb{R}$.

(6) The value of the integral $\int_0^\infty \frac{\cos x}{(1+x^2)^2} dx$ is

A) π/e. B) $\pi/2e$. C) $2\pi e$. D) $2\pi/e$.

(7) If $\sum_{n=-\infty}^{\infty} a_n(z-1)^n$ is the Laurent series expansion of $g(z) = \frac{z^2}{(z-1)^2}$, then a_{-1} is given by

A) 1. B) -1. C) 2. D) $\frac{3}{2}$.

(8) If $\sum_{n=1}^{\infty} a_{2n}z^{2n}$ is the power series of $\frac{z^2}{(1-z^2)^2}$, then a_2 is given by

A) 2. B) 3. C) 6. D) 4.

(9) Let $f(z) = z^2 - 2z$ and $\gamma(t) = (\cos t, \sin t),\ 0 \leq t \leq \pi$. Then $\int_\gamma f(z)dz$ is equal to

A) $-2/3$. B) $1/3$. C) $-1/3$. D) $5/3$.

(10) Let H be a Hilbert space and $\{e_k :\ k \in \mathbb{N}\}$ an orthonormal set in H. Which of the following is true?

A) $\sum_k |\langle x,\ e_k\rangle|^2 = \|x\|^2$ for every $x \in H$.
B) $S = \{e_k : k \in \mathbb{N}\}$ is a Hamel basis for H if S is total in H.
C) $\langle x,\ e_k\rangle = 0$ for all k then $x = 0$.
D) H is separable if $\{e_k : k \in \mathbb{N}\}$ is total in H .

(11) For normed spaces X and Y, let $T : X \to Y$ be a bounded linear operator and dim (Range T) $< \infty$. Which of the following is not true?

A) Ker T is closed in X.
B) $X/\text{Ker } T$ is finite dimensional.
C) dim ($X/\text{Ker } T$) $>$ dim (Range T).
D) Range T is closed.

(12) Let $X = C_{00}$, the linear space of all complex sequences with only a finite number of non-zero entries. For $x = (x_n),\ y = (y_n) \in X$, define an inner product on X by

$$\langle x,\ y\rangle = \sum_{n=1}^{\infty} x_n\overline{y_n}$$

and $f : X \to \mathbb{C}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

Then which of the following is not true?

A) f is bounded with $\|f\| \leq \pi$.
B) there exists $y \in X$ such that $f(x) = \langle x,\ y\rangle$ for all $x \in X$.
C) Ker f is a proper closed subspace of X.
D) $(\text{Ker f})^{\perp} = \{0\}$.

(13) Let $A = \left\{\frac{1}{n} :\ n \in \mathbb{N}\right\}$ and $B = [-1, 1] \setminus A$. If $f : [-1, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in A; \\ -1, & \text{if } x \in B, \end{cases}$$

then f is

A) Riemann integrable.
B) Lebesgue integrable but not Riemann integrable.
C) not Lebesgue integrable.
D) not measurable.

(14) Let $A = \{(x,\ 0) :\ x \in \mathbb{R}\}$ and $B = \{(x, e^{-x}) :\ x \in \mathbb{R}\}$ be subsets of $\mathbb{R}^2$ with Euclidean metric. If $d(A, B)$ denotes the distance between A and B, then A and B are

A) disjoint closed sets in $\mathbb{R}^2$ with $d(A, B) > 0$.
B) disjoint closed sets in $\mathbb{R}^2$ with $d(A, B) = 0$.
C) closed in $\mathbb{R}^2$, $d(A, B) = 0$, but not disjoint.
D) disjoint, $d(A, B) = 0$, but not closed in $\mathbb{R}^2$.

(15) Let V be a Lebesgue non-measurable subset of $\mathbb{R}$ and $p \in \mathbb{R}$ be fixed. Define a function $f_p : \mathbb{R} \to \mathbb{R}$ by

$$f_p(x) = \begin{cases} 2, & x - p \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Then f_p is

A) Lebesgue integrable.
B) not Lebesgue measurable only for $p = 0$.
C) Lebesgue measurable only for $p = 0$.
D) not Lebesgue measurable.

(16) Let $z_i \in \mathbb{C}$ $(i = 1, 2, 3, 4)$ with $z_1 \neq 0$. Which one of the following identities about cross ratio is false?

A) $(z_1,\ z_2,\ z_3,\ z_4) = (z_1 + z_4,\ z_2 + z_4,\ z_3 + z_4,\ 2z_4)$.
B) $(z_1,\ z_2,\ z_3,\ z_4) = (z_1^2,\ z_1z_2,\ z_1z_3,\ z_1z_4)$.
C) $(z_1,\ z_2,\ z_3,\ z_4) = (z_1^2,\ z_2^2,\ z_3^2,\ z_4^2)$.

D) $(z_1,\ z_2,\ z_3,\ z_4) = \left(1,\ \frac{z_2}{z_1},\ \frac{z_3}{z_1},\ \frac{z_4}{z_1}\right)$.

(17) Which of the following is not a reflexive Banach space?

A) For $1 < p < \infty$, $l_p = \{(x_n) :\ \sum |x_n|^p < \infty\}$ with $\|.\|_p$.
B) for $1 \le p \le \infty$, $l_p^n = (\mathbb{R}^n,\ \|.\|_p)$.
C) C_0, the space of sequences converging to zero with sup norm.
D) $C\{1,\ 2,\ \ldots, n\}$, space of real valued continuous functions on $\{1,\ 2,\ \ldots, n\}$ with maximum norm.

(18) Which of the following is not true?

A) If X is uncountable and d is an arbitrary metric on X, then $(X,\ d)$ is separable.
B) Product of two separable metric spaces is separable.
C) $([a,b],d)$ (d is usual metric on $[a,b]$) is separable.
D) For a Lebesgue measurable subset E of $\mathbb{R}$ and $1 \le p \le \infty$, $L^p(E)$ is separable.

(19) Which of the following is not true?

A) $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.
B) For $(x,y) \in \mathbb{R}^2$, the subspace $\mathbb{R}^2 \setminus \{(x,\ y)\}$ is connected and is homeomorphic to $\mathbb{R}$.
C) $\{(x,y) \in \mathbb{R}^2 :\ x = 0,\ -1 \le y \le 1\} \cup \{(x,y) :\ y = \sin\frac{1}{x},\ 0 < x \le 1\}$ is connected.
D) $C = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 = 1\}$ is connected.

(20) Let $f(x) = \sin x$, then which of the following is not true?

A) $\displaystyle\int_0^\infty \frac{f(x)}{x}dx$ exists.
B) $\displaystyle\int_0^\infty \frac{|f(x)|}{x}dx = \pi/2$.
C) $\displaystyle\int_0^\infty \frac{f(x)}{x}dx$ does not exist in $\mathbb{R}$.
D) $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$.

(21) Let H and K be normal solvable subgroups of a group G. Then, HK is

A) solvable but not normal in G.
B) not solvable but normal in G.
C) neither solvable nor normal in G.
D) solvable and normal in G.

(22) The alternating group A_∞ on infinitely many symbols

A) has a proper subgroup of finite index.
B) has no proper subgroup of finite index.
C) not a simple group.

D) none of A), B), C).

(23) Let G be a group of order 26. The number of subgroups of order 5 is

A) 0. B) 5. C) 6. D) 2.

(24) Which one of the following statements is true?

A) An infinite abelian group has a composition series.
B) a finite abelian group has no composition series.
C) a finite abelian group has a composition series.
D) every group has a composition series.

(25) Suppose ED stands for Euclidean domain, PID and UID stands for principal ideal domain and unique factorization domain respectively. Then which of the following statements is true?

A) PID $\Longrightarrow$ UFD $\Longrightarrow$ ED .
B) UFD $\Longrightarrow$ PID $\Longrightarrow$ ED.
C) ED $\Longrightarrow$ PID $\Longrightarrow$ UFD.
D) ED $\Longrightarrow$ UFD $\Longrightarrow$ PID.

(26) Let R be a commutative ring with unity satisfying descending chain condition (d.c.c.) on its ideals. Consider the following statements.
1. R satisfies ascending chain condition (a.c.c.) on its ideals.
2. Every prime ideal in R is maximal.
Which of the following is correct?

A) Statement 1 is correct and statement 2 is not correct.
B) Statement 2 is correct and statement 1 is not correct.
C) Both the statements 1 and 2 are correct.
D) Both the statements 1 and 2 are false.

(27) Let $\mathbb{Z}_p(\alpha)$ be an extension of $\mathbb{Z}_p$ obtained by adjoining α to $\mathbb{Z}_p$, where α is a root of a degree two, irreducible polynomial over $\mathbb{Z}_p$. Then

A) $|\mathbb{Z}_p(\alpha)| = p^3$.
B) $|\mathbb{Z}_p(\alpha)| = p^2$.
C) $|\mathbb{Z}_p(\alpha)| = \infty$.
D) $\mathbb{Z}_p(\alpha)$ is a finite field but cannot say about its number of elements.

(28) Let $F \subseteq E$ be a field extension. Let $\alpha \in E$ be a root of an irreducible polynomial $f(x)$ over F of multiplicity three. Let β be any other root of $f(x)$ in E. Then the multiplicity of β is

A) two. B) greater than three.
C) three. D) one.

(29) The total number of subfields of a field with 729 elements is

A) one. B) two. C) three. D) four.

(30) Let α be a positive constructible number. Then which one of the following is a constructible number?

A) $(\alpha)^{1/3}$. B) $(\alpha)^{1/10}$. C) $(\alpha)^{1/14}$. D) $(\alpha)^{1/16}$.

(31) Let $K = \mathbb{Q}(\sqrt{2}, i)$ be the field generated over $\mathbb{Q}$ by $\sqrt{2}$ and i. Then the dimension of $\mathbb{Q}(\sqrt{2}, i)$, as a $\mathbb{Q}$-vector space is equal to

A) 4. B) 3. C) 2. D) 1.

(32) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Then

A) A and B are diagonalizable.
B) A is diagonalizable but B is not.
C) B is diagonalizable but A is not.
D) neither A nor B is diagonalizable.

(33) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

and let $B = \{(1,2)^t,\ (3,1)^t\}$, $B' = \{(1,0,0)^t,\ (1,1,0)^t,\ (1,1,1)^t\}$ be ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Then the matrix of T relative to B and B' is

A) $\begin{pmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{pmatrix}$. B) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 1 & -2 \end{pmatrix}$.

C) $\begin{pmatrix} -1 & 3 \\ -4 & 2 \\ -1 & 2 \end{pmatrix}$. D) $\begin{pmatrix} 1 & 3 \\ -2 & -4 \\ 1 & -2 \end{pmatrix}$.

(34) Let $B = \{(1,1,0),\ (-1,0,1),\ (0,1,-1)\}$ be a basis of $\mathbb{R}^3$ and its dual basis $B^* = \{f_1,\ f_2,\ f_3\}$, where $f_i,\ i = 1,2,3$ are linear functionals defined on $\mathbb{R}^3$. Then for $x = (x_1, x_2, x_3)$

A) $2f_1(x) = x_1 + x_2 + x_3;\ 2f_2(x) = x_1 - x_2 + x_3;\ 2f_3(x) = -x_1 + x_2 - x_3.$
B) $2f_1(x) = x_1 - x_2 + x_3;\ 2f_2(x) = -x_1 - x_2 + x_3;\ 2f_3(x) = -x_1 + x_2 - x_3.$
C) $2f_1(x) = x_1 + x_2 + x_3;\ 2f_2(x) = -x_1 - x_2 + x_3;\ 2f_3(x) = -x_1 + x_2 + x_3.$
D) $2f_1(x) = x_1 + x_2 + x_3;\ 2f_2(x) = -x_1 + x_2 + x_3;\ 2f_3(x) = -x_1 + x_2 - x_3.$

(35) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$. Then its minimal polynomial is

A) $x^2(x-2)^2$. B) $x^2(x-1)^2$.
C) $(x-1)^2(x-2)^2$. D) $x(x-1)^3$.

(36) The equation $u_{xx} + xu_{yy} = 0$ is

A) elliptic for $x > 0,\ y \in \mathbb{R}$ and hyperbolic for $x < 0,\ y \in \mathbb{R}$.
B) elliptic for all $(x, y) \in \mathbb{R}^2$.
C) hyperbolic for all $(x, y) \in \mathbb{R}^2$.
D) hyperbolic for $x > 0,\ y \in \mathbb{R}$ and elliptic for $x < 0,\ y \in \mathbb{R}$.

(37) Let $G(x, \xi)$ be the Green's function for the linear second order ordinary differential equation $L[y] = f(x)$, where $L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$ with $q, f \in C[a, b]$, $p \in C'[a, b]$ and $p(x) \neq 0$, $x \in [a, b]$. Then which of the following is not true?

A) at the given point $x = \xi \in [a, b]$, the first derivative has jump discontinuity given by $\frac{d}{dx}G(x,\xi)\Big|_{x=\xi^-}^{\xi^+} = \frac{-1}{p(\xi)}$.
B) $G(x, \xi) = G(\xi, x)$.
C) $G(x, \xi)$ and its first and second order derivatives are continuous for all $x \neq \xi$ in $a \leq x,\ \xi \leq b$.
D) for fixed ξ, $G(x, \xi)$ is the solution of the associated homogeneous problem $L[y] = 0$, except at point $x = \xi$.

(38) The initial boundary value problem

$$u_t = u_{xx},\ 0 < x < 1,\ t > 0;$$
$$u(0, t) = u(1, t) = 0;\ u(x, 0) = x(1 - x),\ 0 \leq x \leq 1$$

has solution $u(x, t) = \sum_{n=1}^{\infty} a_n \exp(-n^2\pi t) \sin(n\pi x)$, where a_n is equal to

A) $8/(n^3\pi^3)$ if n is odd and 0 if n is even.
B) $4/(n^3\pi^3)$ if n is odd and 0 if n is even.
C) $8/(n^2\pi^2)$ if n is odd and 0 if n is even.
D) $8/(n^3\pi^3)$ if n is even and 0 if n is odd.

(39) Consider the following partial differential equation

$$z^2(p^2 + q^2 + 1) = r^2$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. Which of the following statements is true about the above equation?

A) $(x-a)^2 + (y-b)^2 + z^2 = r^2$ is a general integral.
B) it is a semi-linear PDE of order 2 and degree 1.
C) $4x^2 + y^2 - 4xy + 5z^2 - 5r^2 = 0$ is a particular solution.
D) A), B), C) are true.

(40) The vertical displacement $u(x, t)$ of an infinitely long elastic string is goverened by the initial value problem

$$u_{tt} = 4u_{xx}, \ -\infty < x < \infty, \ t > 0$$

$$u(x, 0) = -x, \ u_t(x, 0) = 0.$$

The value of $u(x, t)$ at $x = 2, \ t = 2$ is

A) -2. B) -4. C) 2. D) 4.

(41) The equation of the surface satisfying $4yzp + q + 2y = 0, \ p = \frac{\partial z}{\partial x}, \ q = \frac{\partial z}{\partial y}$ and passing through $y^2 + z^2 = 1, \ x + z = 2$ is given by

A) $x^2 + z + z^2 + y^2 = 3$. B) $x + z^2 + z + y^2 = 3$.
C) $y + x + z + z^2 = 3$. D) $x^2 + z + z^2 + y = 3$.

(42) An inviscid incompressible fluid of density ρ moves steadily with velocity $\vec{q} = (kx, -ky, 0)$, where k is constant, under no extremal force. The pressure $p(x, \ y, \ z)$ in the fluid motion when $p(0, 0, 0) = p_0$, is

A) $p_0 - \rho k^2(y^2 - x^2)$. B) $p_0 - \rho k^2(y^2 + x^2)$.
C) $p_0 - \frac{\rho k^2(y^2+x^2)}{2}$. D) $p_0 - \frac{\rho k^2(y^2-x^2)}{2}$.

(43) A sphere is moving with constant velocity U in a liquid which is otherwise at rest. The velocity potential $\phi(r, \theta)$ for the flow is

A) $\frac{1}{2}Ua^3r^{-2}\cos\theta$. B) $\frac{1}{2}Ua^3r\cos\theta$.
C) $\frac{1}{2}Ua^2r^{-2}\cos\theta$. D) none of these.

(44) In two dimensional motion, the vorticity vector is

A) perpendicular to the plane of flow.
B) parallel to the plane of flow.
C) oblique to the plane of flow.

D) may be parallel or perpendicular to the plane of flow.

(45) The singular solution of $p^3 - 4xyp + 8y^2 = 0$, where $p = \frac{dy}{dx}$ is

A) $27y - 4x^2 = 0$.
B) $27y - 4x^3 = 0$.
C) $27y + 4x^3 = 0$.
D) $27y + 4x^2 = 0$.

(46) The arrangement of sources and sinks for $w = \log\left(z - \frac{a^2}{z}\right)$ are

A) a source of unit strength at $(a, 0)$.
B) two sinks of unit strength at $(\pm a^2, 0)$.
C) a source of unit strength at origin and two sinks of unit strength at $(a, 0)$ and $(-a, 0)$.
D) a source of unit strength at $(a, 0)$ and a sink at $(0, -a)$ of unit strength.

(47) By the method of variation of parameters, the particular solution of the equation $x^2y'' + xy' - y = x^2e^x$ is

A) $e^x + \frac{1}{x}$.
B) $e^x - \frac{e^x}{x}$.
C) $e^x + \frac{e^x}{x}$.
D) $e^x - \frac{1}{x}$.

(48) The solution of $u_{xx} - 4u_{xy} + 4u_{yy} = 0$ is

A) $u = f(y + 2x) + g(y + 2x)$.
B) $u = f(y - 2x) + g(y + 2x)$.
C) $u = f(y + 2x) + xg(y + 2x)$.
D) $u = f(y + 2x) - xg(y + 2x)$.

(49) A complete integral of $f = xz_xz_y + yz_x^2 - 1 = 0$ obtained by Charpit's method is

A) $(z + b)^2 = 4(ax - y)$.
B) $(z + b)^2 = 4(ax + y)$.
C) $(z - b)^2 = 4(ax + y)$.
D) $(z + b)^2 = 4ax + y^2$.

(50) The tensor form of Navier Stokes equations with x_i as space coordinates, u_i as velocity components, F_i as components of body forces, Δ as rate of dilation, p as pressure, ρ as density and ν as coefficient of kinematic viscosity, is given by

A) $\frac{du_i}{dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + \frac{2}{3}\nu\frac{\partial \Delta}{\partial x_i}$.
B) $\frac{du_i}{dt} = F_i + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + \frac{2}{3}\nu\frac{\partial \Delta}{\partial x_i}$.
C) $\frac{du_i}{dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial \Delta}{\partial x_i}$.
D) $\frac{du_i}{dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + \frac{2}{3}\nu\Delta$.