M.A./M.Sc. (Mathematics) Entrance Examination 2016-17

Max Time: 2 hours — Max Marks: 150

Instructions: There are 50 questions. Every question has four choices of which exactly one is correct. *For correct answer, 3 marks will be given. For wrong answer, 1 mark will be deducted.* Scientific calculators are allowed.

In the following $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of all real numbers, natural numbers, rational numbers, integers and complex numbers respectively.

(1) Let X be a countably infinite subset of $\mathbb{R}$ and A be a countably infinite subset of X. Then the set $X \setminus A = \{x \in X \mid x \notin A\}$

A) is empty. B) is a finite set . C) can be a countably infinite set. D) can be an uncountable set.

(2) The subset $A = \{x \in \mathbb{Q} : x^2 < 4\}$ of $\mathbb{R}$ is

A) bounded above but not bounded below.
B) bounded above and $\sup A = 2$.
C) bounded above but does not have a supremum .
D) not bounded above .

(3) Let f be a function defined on $[0, \infty)$ by $f(x) = [x]$, the greatest integer less than or equal to x. Then

A) f is continuous at each point of $\mathbb{N}$.
B) f is continuous on $[0, \infty)$.
C) f is discontinuous at $x = 1, 2, 3, \ldots$.
D) f is continuous on $[0, 7]$.

(4) The series $x + \dfrac{2^2x^2}{2!} + \dfrac{3^3x^3}{3!} + \cdots$ is convergent if x belongs to the interval

A) $(0, 1/e)$. B) $(1/e, \infty)$. C) $(2/e, 3/e)$. D) $(3/e, 4/e)$.

(5) The subset $A = \{x \in \mathbb{Q} : -1 < x < 0\} \cup \mathbb{N}$ of $\mathbb{R}$ is

A) bounded, infinite set and has a limit point in $\mathbb{R}$.
B) unbounded, infinite set and has a limit point in $\mathbb{R}$.
C) unbounded, infinite set and does not have a limit point in $\mathbb{R}$.
D) bounded, infinite set and does not have a limit point in $\mathbb{R}$.

(6) Let f be a real-valued monotone non-decreasing function on $\mathbb{R}$. Then

A) for $a \in \mathbb{R}$, $\lim\limits_{x \to a} f(x)$ exists .
B) f is an unbounded function.

C) $h(x) = e^{-f(x)}$ is a bounded function.
D) if $a < b$, then $\lim_{x \to a^+} f(x) \leq \lim_{x \to b^-} f(x)$.

(7) Let $X = C[0,1]$ be the space of all real-valued continuous functions on $[0,1]$. Then (X, d) is not a complete metric space if

A) $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. B) $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$.

C) $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. D) $d(f,g) = \begin{cases} 0, & \text{if } f = g \\ 1, & \text{if } f \neq g \end{cases}$.

(8) The series $\sum_{k=0}^{\infty} \frac{k^2 + 3k + 1}{(k+2)!}$ converges to

A) 1. B) 1/2. C) 2. D) 3.

(9) We know that $xe^x = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$. The series $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$ converges to

A) e^2. B) $2\,e^2$. C) $4\,e^2$. D) $6\,e^2$.

(10) Let $X = \mathbb{R}^2$ with metric defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$. Then

A) every subset of X is dense in (X,d).
B) (X,d) is separable .
C) (X,d) is compact but not connected.
D) every subspace of (X,d) is complete.

(11) Let d_1 and d_2 be metrics on a non-empty set X. Which of the following is not a metric on X?

A) $\max(d_1, d_2)$. B) $\sqrt{d_1^2 + d_2^2}$. C) $1 + d_1 + d_2$. D) $\frac{1}{4}d_1 + \frac{3}{4}d_2$.

(12) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \sqrt{|xy|}$. Then at origin

A) f is continuous and $\frac{\partial f}{\partial x}$ exists .
B) f is discontinuous and $\frac{\partial f}{\partial x}$ exists.
C) f is continuous but $\frac{\partial f}{\partial x}$ does not exist .
D) f is discontinuous but $\frac{\partial f}{\partial x}$ exists.

(13) The sequence of real-valued functions $f_n(x) = x^n$, $x \in [0,1] \cup \{2\}$, is

A) pointwise convergent but not uniformly convergent.
B) uniformly convergent.

C) bounded but not pointwise convergent.
D) not bounded.

(14) The integral $\int_0^\infty \sin x\, dx$

A) exists and equals 0.
B) exists and equals 1.
C) exists and equals -1.
D) does not exist.

(15) If $\{a_n\}$ is a bounded sequence of real numbers, then

A) $\inf_n a_n \leq \liminf_{n\to\infty} a_n$ and $\sup_n a_n \leq \limsup_{n\to\infty} a_n$.
B) $\liminf_{n\to\infty} a_n \leq \inf_n a_n$.
C) $\liminf_{n\to\infty} a_n \leq \inf_n a_n$ and $\sup_n a_n \leq \limsup_{n\to\infty} a_n$.
D) $\inf_n a_n \leq \liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n \leq \sup_n a_n$.

(16) The series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

A) diverges .
B) converges to 1.
C) converges to $\frac{1}{2}$.
D) converges to $\frac{1}{7}$.

(17) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y}, & x^2 \neq -y \\ 0, & x^2 = -y. \end{cases}$$

Then

A) directional derivative does not exist at $(0,0)$.
B) f is continuous at $(0,0)$.
C) f is differentiable at $(0,0)$.
D) each directional derivative exists at $(0,0)$ but f is not continuous.

(18) Let $f : [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and F be its indefinite integral. Which of the following is not true?

A) $F'(0)$ does not exist.
B) F is an anti-derivative of f on $[-1,1]$.
C) F is Riemann integrable on $[-1,1]$.

D) F is continuous on $[-1,1]$.

(19) Let $f(x)=x^2$, $x\in[0,1]$. For each $n\in\mathbb{N}$, let P_n be the partition of $[0,1]$ given by $P_n=\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n}{n}\}$. If $\alpha_n=U(f,P_n)$ (upper sum) and $\beta_n=L(f,P_n)$ (lower sum) then

A) $\alpha_n=(n+2)(2n+1)/(6n^2)$.
B) $\beta_n=(n-2)(2n+1)/(6n^2)$.
C) $\beta_n=(n-1)(2n-1)/(6n^2)$.
D) $\lim_{n\to\infty}\alpha_n\neq\lim_{n\to\infty}\beta_n$.

(20) Let $I=\displaystyle\int_0^{\pi/2}\log\sin x\,dx$. Then

A) I diverges at $x=0$.
B) I converges and is equal to $-\pi\log 2$.
C) I converges and is equal to $-\frac{\pi}{2}\log 2$.
D) I diverges at $x=\frac{\pi}{4}$.

(21) Which of the following polynomials is not irreducible over $\mathbb{Z}$?

A) $x^4+125x^2+25x+5$.
B) $2x^3+6x+12$.
C) x^3+2x+1.
D) $x^4+x^3+x^2+x+1$.

(22) A complex number α is said to be algebraic integer if it satisfies a monic polynomial equation with integer coefficients. Which of the following is not an algebraic integer?

A) $\sqrt{2}$.
B) $\frac{1}{\sqrt{2}}$.
C) $\frac{1-\sqrt{5}}{2}$.
D) $\sqrt{\alpha}$, α is an algebraic integer.

(23) If $A=\begin{bmatrix}1&1&1\\0&2&-1\\0&0&-2\end{bmatrix}$, then the value of $A^4-A^3-4A^2+4I$ is

A) $4\begin{bmatrix}-2&1&0\\0&-3&1\\0&0&1\end{bmatrix}$.
B) $4\begin{bmatrix}0&-1&-1\\0&-1&1\\0&0&3\end{bmatrix}$.
C) $4\begin{bmatrix}2&1&1\\0&3&-1\\0&0&-1\end{bmatrix}$.
D) $4\begin{bmatrix}0&1&1\\0&1&-1\\0&0&-3\end{bmatrix}$.

(24) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (x+y, x-y, 2y)$. If $\{(1,1),(1,0)\}$ and $\{(1,1,1),(1,0,1),(0,0,1)\}$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, then the matrix representation of T with respect to the ordered bases is

A) $\begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$. B) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$.

C) $\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 0 & -1 \end{bmatrix}$. D) $\begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$.

(25) Let P_4 be real vector space of real polynomials of degree ≤ 4. Define an inner product on P_4 by

$$\left\langle \sum_{i=0}^{4} a_i x^i, \sum_{i=0}^{4} b_i x^i \right\rangle = \sum_{i=0}^{4} a_i b_i.$$

Then the set $\{1, x, x^2, x^3, x^4\}$ is

A) orthogonal but not orthonormal .
B) orthonormal .
C) not orthogonal.
D) none of these.

(26) If $\{a + ib, c + id\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$, then

A) $ac - bd = 0$. B) $ac - bd \neq 0$.
C) $ad - bc = 0$. D) $ad - bc \neq 0$.

(27) Consider $M_1 = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$, $M_2 = \begin{pmatrix} -1 & 2 \\ 3 & 2 \end{pmatrix}$, $M_3 = \begin{pmatrix} 5 & -6 \\ -3 & -2 \end{pmatrix}$ and $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ of $M_{2\times 2}(\mathbb{R})$. Then

A) $\{M_2, M_3, M_4\}$ is linearly independent.
B) $\{M_1, M_2, M_4\}$ is linearly independent.
C) $\{M_1, M_3, M_4\}$ is linearly independent.
D) $\{M_1, M_2, M_3\}$ is linearly dependent.

(28) If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \alpha M_1 + \beta M_2 + \gamma M_3$, where $M_1 = I_{2\times 2}$, $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $M_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then

A) $\alpha = \beta = 1,\ \gamma = 2.$ B) $\alpha = \beta = -1,\ \gamma = 2.$
C) $\alpha = 1,\ \beta = -1,\ \gamma = 2.$ D) $\alpha = -1,\ \beta = 1,\ \gamma = 2.$

(29) Let W be the subset of the vector space $V = M_{n\times n}(\mathbb{R})$ consisting of symmetric matrices. Then

A) W is not a subspace of V.
B) W is a subspace of V of dimension $\frac{n(n-1)}{2}$.
C) W is a subspace of V of dimension $\frac{n(n+1)}{2}$.
D) W is a subspace of V of dimension $n^2 - n$.

(30) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation and B be a basis of $\mathbb{R}^3$ given by $B = \{(1,1,1)^t, (1,2,3)^t, (1,1,2)^t\}$. If $T\big((1,1,1)^t\big) = (1,1,1)^t$, $T\big((1,2,3)^t\big) = (-1,-2,-3)^t$ and $T\big((1,1,2)^t\big) = (2,2,4)^t$ (A^t being the transpose of A), then $T\big((2,3,6)^t\big)$ is

A) $(2,1,4)^t$. B) $(1,2,4)^t$.
C) $(3,2,1)^t$. D) $(2,3,4)^t$.

(31) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation and $B = \{v_1, v_2, v_3\}$ be a basis for $\mathbb{R}^3$. Suppose that $T(v_1) = (1,1,0)^t$, $T(v_2) = (1,0,-1)^t$ and $T(v_3) = (2,1,-1)^t$ then

A) $w = (1,2,1)^t \notin$ Range of T .
B) $\dim$ (Range of T) = 1.
C) $\dim$ (Null space of T) = 2.
D) Range of T is a plane in $\mathbb{R}^3$.

(32) The last two digits of the number $9^{(9^9)}$ is

A) 29. B) 89. C) 49. D) 69.

(33) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ under matrix multiplication, where $ad - bc \neq 0$ and a, b, c, d are integers modulo 3. The order of G is

A) 24. B) 16. C) 48. D) 81.

(34) For the ideal $I = \langle x^2 + 1 \rangle$ of $\mathbb{Z}[x]$, which of the following is true?

A) I is a maximal ideal but not a prime ideal.
B) I is a prime ideal but not a maximal ideal.
C) I is neither a prime ideal nor a maximal ideal.
D) I is both prime and maximal ideal.

(35) Consider the following statements:

1. Every Euclidean domain is a principal ideal domain;
2. Every principal ideal domain is a unique factorization domain;
3. Every unique factorization domain is a Euclidean domain.

Then

A) statements 1 and 2 are true.
B) statements 2 and 3 are true.
C) statements 1 and 3 are true.
D) statements 1, 2 and 3 are true.

(36) The ordinary differential equation:

$$\frac{dy}{dx} = \frac{2y}{x}$$

with the initial condition $y(0) = 0$, has

A) infinitely many solutions.
B) no solution.
C) more than one but only finitely many solutions.
D) unique solution.

(37) Consider the partial differential equation:

$$4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} - 9u = 9.$$

Which of the following is not correct?

A) It is a second order parabolic equation.
B) The characteristic curves are given by $\zeta = 2y - 3x$ and $\eta = y$.
C) The canonical form is given by $\dfrac{\partial^2 u}{\partial \eta^2} - u = 1$, where η is a characteristic variable.
D) The canonical form is $\dfrac{\partial^2 u}{\partial \eta^2} + u = 1$, where η is a characteristic variable.

(38) Consider the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},\ x > 0,\ t > 0$$

subject to the initial conditions:

$$u(x, 0) = |\sin x|,\ x \geq 0$$
$$u_t(x, 0) = 0,\ x \geq 0$$

and the boundary condition:

$$u(0, t) = 0,\ t \geq 0.$$

Then $u\left(\pi, \dfrac{\pi}{4}\right)$ is equal to

A) 1. B) 0. C) $\frac{1}{2}$. D) $-\frac{1}{2}$.

(39) The initial value problem

$$x\frac{dy}{dx} = y + x^2,\ x > 0,\ y(0) = 0$$

has

A) infinitely many solutions. B) exactly two solutions.
C) a unique solution. D) no solution.

(40) In a tank there is 120 litres of brine (salted water) containing a total of 50 kg of dissolved salt. Pure water is allowed to run into the tank at the rate of 3 litres per minute. Brine runs out of the tank at the rate of 2 litres per minute. The instantaneous concentration in the tank is kept uniform by stirring. How much salt is in the tank at the end of one hour?

A) 15.45 kg. B) 19.53 kg. C) 14.81 kg. D) 18.39 kg.

(41) If the differential equation

$$2t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 3y = 0$$

is associated with the boundary conditions $y(1) = 5$, $y(4) = 9$, then $y(9) =$

A) 27.44. B) 13.2. C) 19. D) 11.35.

(42) The third degree hermite polynomial approximation for the function $y = y(x)$ such that $y(0) = 1$, $y'(0) = 0$, $y(1) = 3$ and $y'(1) = 5$ is given by

A) $1 + x^2 + x^3$. B) $1 + x^3 + x$.
C) $x^2 + x^3$. D) none of the above.

(43) Let y be the solution of the initial value problem

$$\frac{dy}{dx} = y - x,\ y(0) = 2.$$

Using Runge-Kutta second order method with step size $h = 0.1$, the approximate value of $y(0.1)$ correct to four decimal places is given by

A) 2.8909. B) 2.7142. C) 2.6714. D) 2.7716.

(44) Consider the system of linear equations

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}.$$

With the initial approximation $[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]^T = [0,0,0]^T$, the approximate value of the solution $[x_1^{(1)}, x_2^{(1)}, x_3^{(1)}]^T$ after one iteration by Gauss Seidel method is

A) $[3.2, 2.25, 1.5]^T$.
B) $[3.5, 2.25, 1.625]^T$.
C) $[2.25, 3.5, 1.625]^T$.
D) $[2.5, 3.5, 1.6]^T$.

(45) For the wave equation

$$u_{tt} = 16\, u_{xx},$$

the characteristic coordinates are

A) $\xi = x + 16\,t,\ \eta = x - 16\,t$.
B) $\xi = x + 4\,t,\ \eta = x - 4\,t$.
C) $\xi = x + 256\,t,\ \eta = x - 256\,t$.
D) $\xi = x + 2\,t,\ \eta = x - 2\,t$.

(46) Let f_1 and f_2 be two solutions of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0,$$

where a_0, a_1 and a_2 are continuous on $[0,1]$ and $a_0(x) \neq 0$ for all $x \in [0,1]$. Moreover, let $f_1\left(\frac{1}{2}\right) = f_2\left(\frac{1}{2}\right) = 0$. Then

A) one of f_1 and f_2 must be identically zero.
B) $f_1(x) = f_2(x)$ for all $x \in [0,1]$.
C) $f_1(x) = c\, f_2(x)$ for some constant c.
D) none of the above.

(47) The Laplace transform of $e^{4\,t}$ is

A) $1/(s+2)$.
B) $1/(s-2)$.
C) $1/(s+4)$.
D) $1/(s-4)$.

(48) Let $f(t) = 4\sin^2 t$ and let $\sum_{n=0}^{\infty} a_n \cos n\,t$ be the Fourier cosine series of $f(t)$. Which one is true?

A) $a_0 = 0,\ a_2 = 1,\ a_4 = 2$.
B) $a_0 = 2,\ a_2 = 0,\ a_4 = -2$.
C) $a_0 = 2,\ a_2 = -2,\ a_4 = 0$.
D) $a_0 = 0,\ a_2 = -2,\ a_4 = 2$.

(49) For a, b, $c \in \mathbb{R}$, if the differential equation

$$(ax^2 + bxy + y^2)dx + (2x^2 + cxy + y^2)dy = 0$$

is exact, then

A) $b = 2,\ c = 2\,a$.
B) $b = 4,\ c = 2$.
C) $b = 2,\ c = 4$.
D) $b = 2,\ a = 2\,c$.

(50) Let $u(x, t)$ be the solution of the wave equation

$$u_{xx} = u_{tt}, \quad u(x, 0) = \cos(5\pi x), \quad u_t(x, 0) = 0.$$

Then the value of $u(1, 1)$ is

A) -1.
B) 0.
C) 2.
D) 1.