CSIR NET/JRF

Mathematical Science

26 Nov. 2020

PART-A

(Mathematical Sciences)

(1.) Find the value of $f(0)$ is $f(x+2)=(x+1)^{34}-(x+1)^{33}+5$

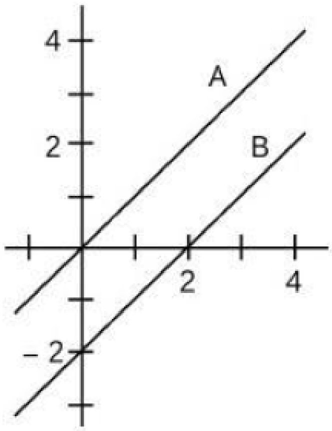
(a.) 5

(b.) 7

(c.) 6

(d.) 72

(2.) The shortest distance between the parallel lines A and B in the following figure is



(a.) $\sqrt{2}$

(b.) 2

(c.) $2\sqrt{2}$

(d.) $2\sqrt{3}$

(3.) Two varieties A and B of rice cost Rs. 30 and Rs. 90 per kg, whereas two varieties C and D of pulses, Rs. 100 and Rs. 120 per kg, respectively. If at least one kg each of A and B and at least half a kg each of C and D have to be purchased, then the minimum and maximum costs of a total of 5 kg of these provisions are, respectively

(a.) Rs. 150 and Rs. 600

(b.) Rs. 260 and Rs. 530

(c.) Rs. 290 and Rs. 470

(d.) Rs. 370 and Rs. 460

(4.) One of four suspects A, B, C and D has committed a crime. A and D are always truthful, and B and C are always untruthful. C and D are identical twins and the interrogator does not know who is who. If A says, " D is innocent", B says, " A is guilty" and among C and D one say, " A is innocent" and the other say, " B is guilty", then which of the following is FALSE?

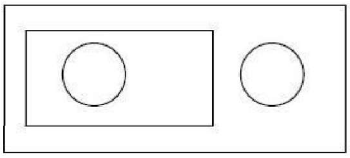
(a.) D said " A is innocent"

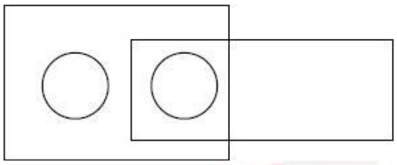
(b.) D is innocent

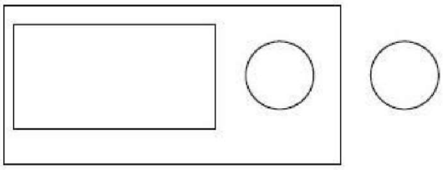
(c.) B is innocent

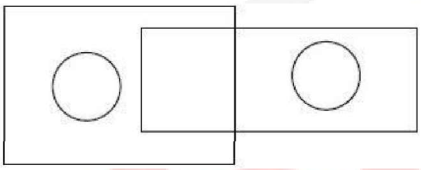
(d.) C is innocent

(5.) Which is an appropriate diagram to represent the relations between the following categories : quadruped, mammal, whale, house lizard?

(a.) 

(b.) 

(c.) 

(d.) 

(6.) A 7m long tube having inner diameter of 2 cm is filled with water. The water is the poured into a cylindrical bucket having inner base area of 200 cm^2. What will be the approximate height (in cm) of water in the bucket?

(a.) 22

(b.) 44

(c.) 9

(d.) 11

(7.) Water is being filled in a cone from the top at a constant volumetric rate. The rate of increase of the height of the water column

(a.) Is linearly dependent on time

(b.) Depends on the apex angle of the cone

(c.) Increases as cube-root of the volumetric rate

(d.) Increases as square-root of the volumetric rate

(8.) A square board is divided into 9 smaller identical squares by drawing lines. Three bullets are shot at the board randomly. The probability that at least 2 bullets hit the same small square is,

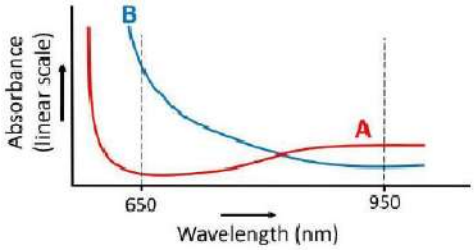
(a.) 1/3

(b.) 56/81

(c.) 25/81

(d.) 2/3

(9.) The wavelength dependent absorbance of two compounds A and B, is show. Absorbance of mixture is a linear function of the concentration of the two compounds. R is defined as a ratio of absorbance at 650 nm, to the absorbance at 950 nm.



If the mixture contains 95% of compound A then R must be

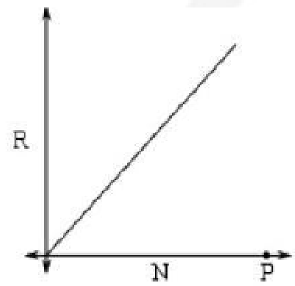
(a.) 95

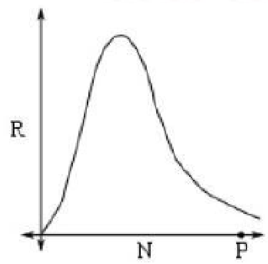
(b.) 5

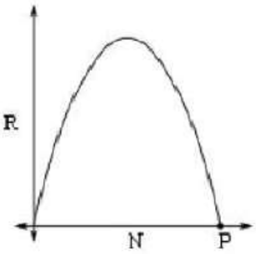
(c.) 1

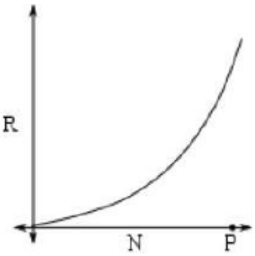
(d.) Less than 1

(10.) An epidemic is spreading in a population of size P. The rate of spread R of the disease at a given time is proportional to both, the number of people affected by the disease (N), and the number of people not yet affected by the disease. Which of the following graphs of R vs N is correct?

(a.) 

(b.) 

(c.) 

(d.) 

(11.) A and B complete a work in 30 days. B and C complete the same work in 24 days whereas C and A complete the same work in 28 days. Based on these statements which of the following conclusions is correct?

(a.) C is the most efficient and B is the least efficient

(b.) B is the most efficient but, the least efficient one cannot be determined

(c.) C is the most efficient but, the least efficient one cannot be determined.

(d.) C is the most efficient and A is the least efficient

(12.) Clock A loses 4 minutes every hour, clock B always shows the correct time and clock C gains 3 minutes every hour. On a Monday, all the three clocks showed the same time, 8 pm. On the following Wednesday, when the clock C shows 2 pm, what time will clock A show?

(a.) 7 : 20 am

(b.) 8 : 40 am

(c.) 9 : 20 am

(d.) 10 : 40 am

(13.) In a class, there is one pencil for every two students, one eraser for every three students, and one ruler for every four students. If the total number of these stationery items required is 65, how many students are present in the class?

(a.) 55

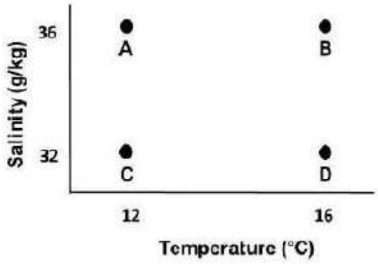
(b.) 60

(c.) 65

(d.) 70

(14.) The figure shows temperature and salinity of four samples of water.

Which one of the samples has the highest density?



(a.) A
(b.) B
(c.) C
(d.) D

(15.) The given table shows the numbers of active and recovered cases of a certain disease. Assuming that the linear trend for both continues, on which day will recovered cases be twice that of the active cases?

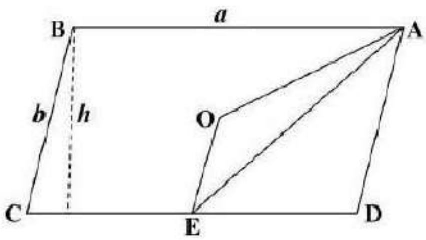
Day	0	1	4	7	10
Active cases	990	1000	1030	1060	1090
Recovered cases	760	800	920	1040	1160

(a.) 61
(b.) 62
(c.) 63
(d.) 64

(16.) A boat weighs 60 kg, and oarsmen A and B weigh 80 and 90 kg, respectively. Rowing at a constant power, the time required to complete a course is proportional to the total weight. Rowing alone, A and B complete the course in 1 and $1\frac{1}{2}$ hours, respectively. Assuming that their powers add up, how long will they take to complete the course if they row together?

(a.) 49.5 min
(b.) 57.5 min
(c.) 62.6 min
(d.) 72.5 min

(17.) Consider a parallelogram $ABCD$ with centre O and E as the midpoint of side CD. The area of the triangle OAE, is



(a.) $\frac{1}{5}ah$

(b.) $\frac{1}{6}ah$

(c.) $\frac{1}{8}ah$

(d.) $\frac{1}{7}ah$

(18.) The sum of the first n even numbers is

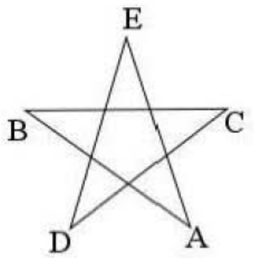
(a.) Divisible by n and not by $(n+1)$

(b.) Divisible by $(n+1)$ and not by n

(c.) Divisible by both n and $(n+1)$

(d.) Neither divisible by n nor by $(n+1)$

(19.) A, B, C, D and E are the vertices of a regular pentagon as shown in the figure.



The angle $\angle ABC$ is

(a.) 48°

(b.) 72°

(c.) 54°

(d.) 36°

(20.) On a 200 m long straight road, maximum number of poles are fixed at 20 m interval. How many of these poles should be removed in order to have maximum number of poles at an interval of 40 m on the road?

(a.) 8

(b.) 6

(c.) 5

(d.) 4

PART-B
(Mathematical Sciences)

(21.) Let $\{E_n\}$ be a sequence of subsets of $\mathbb{R}$.

Define $\limsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$

$\liminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$

Which of the following statements is true?

(a.) $\limsup_n E_n = \liminf_n E_n$

(b.) $\limsup_n E_n = \{x : x \in E_n \text{ for some } n\}$

(c.) $\liminf_n E_n = \{x : x \in E_n \text{ for all finitely many } n\}$

(d.) $\liminf_n E_n = \{x : x \in E_n \text{ for infinitely many } n\}$

(22.) $f : \mathbb{N} \to \mathbb{N}$ be a bounded function. Which of the following statements is NOT true?

(a.) $\limsup_{n\to\infty} f(n) \in \mathbb{N}$

(b.) $\liminf_{n\to\infty} f(n) \in \mathbb{N}$

(c.) $\liminf_{n\to\infty} (f(n)+n) \in \mathbb{N}$

(d.) $\limsup_{n\to\infty} (f(n)+n) \notin \mathbb{N}$

(23.) Which of the following statements is true?

(a.) There are at most countably many continuous maps from $\mathbb{R}^2$ to $\mathbb{R}$

(b.) There are at most finitely many continuous surjective maps from $\mathbb{R}^2$ to $\mathbb{R}$

(c.) There are infinitely many continuous injective maps from $\mathbb{R}^2$ to $\mathbb{R}$

(d.) There are no continuous bijective maps from $\mathbb{R}^2$ to $\mathbb{R}$

(24.) The series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^{\log_e n}}, x \in \mathbb{R}$ converges

(a.) Only for $x = 0$

(b.) Uniformly only for $x \in [-\pi, \pi]$

(c.) Uniformly only for $x \in \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$

(d.) Uniformly for all $x \in \mathbb{R}$

(25.) Given $(a_n)_{n\geq 1}$ a sequence of real numbers, which of the following statements is true?

(a.) $\sum_{n\geq 1}(-1)^n \frac{a_n}{1+|a_n|}$ converges

(b.) There is a subsequence $(a_{n_k})_{k\geq 1}$ such that $\sum_{n\geq 1}\frac{a_{n_k}}{1+|a_{n_k}|}$ converges

(c.) There is a number b such that $\sum_{n\geq 1}\left|b-\frac{a_n}{1+|a_n|}\right|(-1)^n$ converges

(d.) There is a number b and a subsequence $(a_{n_k})_{k\geq 1}$ such that $\sum_{n\geq 1}\left|b-\frac{a_{n_k}}{1+|a_{n_k}|}\right|$ converges

(26.) Given f, g are continuous functions on $[0,1]$ such that $f(0)=f(1)=0$; $g(0)=g(1)=1$ and $f(1/2)>g(1/2)$. Which of the following statements is true?

(a.) There is no $t\in[0,1]$ such that $f(t)=g(t)$

(b.) There is exactly one $t\in[0,1]$ such that $f(t)=g(t)$

(c.) There are at least two $t\in[0,1]$ such that $f(t)=g(t)$

(d.) There are always infinitely many $t\in[0,1]$ such that $f(t)=g(t)$

(27.) Let A be an $n\times n$ matrix such that the set of all its non-zero eigenvalues has exactly r elements. Which of the following statements is true?

(a.) Rank $A\leq r$

(b.) If $r=0$, then rank $A<n-1$

(c.) Rank $A\geq r$

(d.) A^2 has r distinct non-zero eigenvalues

(28.) Let A and B be 2×2 matrices. Then which of the following is true?

(a.) $\det(A+B)+\det(A-B)=\det A+\det B$

(b.) $\det(A+B)+\det(A-B)=2\det A-2\det B$

(c.) $\det(A+B)+\det(A-B)=2\det A+2\det B$

(d.) $\det(A+B)-\det(A-B)=2\det A-2\det B$

(29.) If $A=\begin{pmatrix}3 & -2\\ 2 & -1\end{pmatrix}$, then A^{20} equals

(a.) $\begin{pmatrix}41 & 40\\ -40 & -39\end{pmatrix}$

(b.) $\begin{pmatrix}41 & -40\\ 40 & -39\end{pmatrix}$

(c.) $\begin{pmatrix}41 & -40\\ -40 & -39\end{pmatrix}$

(d.) $\begin{pmatrix} 41 & 40 \\ 40 & -39 \end{pmatrix}$

(30.) Let A be a 2×2 real matrix with $\det A = 1$ and trace $A = 3$. What is the value of trace A^2 ?

(a.) 2
(b.) 10
(c.) 9
(d.) 7

(31.) For $a,b\in\mathbb{R}$, let $p(x,y)=a^2x_1y_1+abx_2y_1+abx_1y_2+b^2x_2y_2$, $x=(x_1,x_2)$, $y=(y_1,y_2)\in\mathbb{R}^2$. For what values of a and b does $p:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ define an inner product?

(a.) $a>0, b>0$
(b.) $ab>0$
(c.) $a=0, b=0$
(d.) For no values of a, b

(32.) Which of the following real quadratic forms on $\mathbb{R}^2$ is positive definite?

(a.) $Q(X,Y)=XY$
(b.) $Q(X,Y)=X^2-XY+Y^2$
(c.) $Q(X,Y)=X^2+2XY+Y^2$
(d.) $Q(X,Y)=X^2+XY$

(33.) Let γ be the positively oriented circle in the complex plane given by $\{z\in\mathbb{C}:|z-1|=1\}$. Then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^3-1}$ equals

(a.) 3
(b.) 1/3
(c.) 2
(d.) 1/2

(34.) For a positive integer p, consider the holomorphic function $f(z)=\frac{\sin z}{z^p}$ for $z\in\mathbb{C}\setminus\{0\}$.

For which values of p does there exist a holomorphic function $g:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ such that $f(z)=g'(z)$ for $z\in\mathbb{C}\setminus\{0\}$?

(a.) All even integers
(b.) All odd integers
(c.) All multiples of 3
(d.) All multiples of 4

(35.) Let γ be the positively oriented circle in the complex plane given by $\{z \in \mathbb{C} : |z-1| = 1/2\}$. The line integral $\int_\gamma \frac{ze^{1/z}}{z^2-1} dz$ equals

(a.) $i\pi e$

(b.) $-i\pi e$

(c.) πe

(d.) $-\pi e$

(36.) Let p be a positive integer. Consider the closed curve $r(t) = e^{it}, 0 \le t < 2\pi$. Let f be a function holomorphic in $\{z : |z| < R\}$ where $R > 1$. If f has a zero only at z_0, $0 < |z_0| < R$, and it is of multiplicity q, then $\frac{1}{2\pi i}\int_r \frac{f'(z)}{f(z)} z^p dz$ equals

(a.) qz_0^p

(b.) $z_0 q^p$

(c.) pz_0^q

(d.) $z_0 p^q$

(37.) Which of the following statements is true?

(a.) Every even integer $n \ge 16$ divides $(n-1)!+3$

(b.) Every odd integer $n \ge 16$ divides $(n-1)!$

(c.) Every even integer $n \ge 16$ divides $(n-1)!$

(d.) For even integer $n \ge 16$, n^2 divides $n!+1$

(38.) Let X be a non-empty set and $P(X)$ be the set of all subsets of X. On $P(X)$, define two operations $\bigstar$ and Δ as follows: for $A, B \in P(X)$, $A \bigstar B = A \cap B$; $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Which of the following statements is true?

(a.) $P(X)$ is a group under $\bigstar$ as well as under Δ

(b.) $P(X)$ is a group under $\bigstar$, but not under Δ

(c.) $P(X)$ is a group under Δ, but not under $\bigstar$

(d.) $P(X)$ is neither a group under $\bigstar$ nor under Δ

(39.) Let $\varphi(n)$ be the cardinality of the set $\{a \mid 1 \le a \le n, (a, n) = 1\}$ where (a, n) denotes the gcd of a and n. Which of the following is NOT true?

(a.) There exist infinitely many n such that $\varphi(n) > \varphi(n+1)$

(b.) There exist infinitely many n such that $\varphi(n) < \varphi(n+1)$

(c.) There exists $N \in \mathbb{N}$ such that $N > 2$ and for all $n > N$, $\varphi(N) < \varphi(n)$

(d.) The set $\left\{\frac{\varphi(n)}{n} : n \in \mathbb{N}\right\}$ has finitely many limit points

(40.) For any two metric spaces $(X, d_X), (Y, d_Y)$ a map $f : X \to Y$ is said to be a closed map if whenever F is closed in X, then $f(F)$ is closed in Y. For any subset B of a metric space, B is given the induced metric. The metric on $X \times Y$ is given by $d((x, y), (x', y'))$ $= \max\{d_X(x, x'), d_Y(y, y')\}$. Which of the following are true?

(a.) For any subset $A \subseteq X$ the inclusion map $i : A \to X$ is closed

(b.) The projection map $p_1 : X \times Y \to X$ given by $p_1(x, y) = x$ is closed

(c.) Suppose that $f : X \to Y, g : Y \to Z$ are continuous maps. If $g \circ f : X \to Z$ is a closed map then $g|_{f(X)} : f(X) \to Z$ is closed. Here $g|_{f(X)}$ means the map g restricted to $f(X)$

(d.) If $f : X \to Y$ takes closed balls into closed sets then f is closed

(41.) Let k be a positive integer. Consider the differential equation

$$\begin{cases} \frac{dy}{dt} = y^{\frac{5k}{5k+2}} \text{ for } t > 0, \\ y(0) = 0 \end{cases}$$

Which of the following statements is true?

(a.) It has a unique solution which is continuously differentiable on $(0, \infty)$

(b.) It has at most two solutions which are continuously differentiable on $(0, \infty)$

(c.) It has infinitely many solutions which are continuously differentiable on $(0, \infty)$

(d.) It has no continuously differentiable solution on $(0, \infty)$

(42.) Let $y_0 > 0, z_0 > 0$ and $\alpha > 1$.

Consider the following two differential equations:

$$(*)\begin{cases} \frac{dy}{dt} = y^{\alpha} \text{ for } t > 0, \\ y(0) = y_0 \end{cases}$$

$$(**)\begin{cases} \frac{dz}{dt} = -z^{\alpha} \text{ for } t > 0, \\ z(0) = z_0 \end{cases}$$

We say that the solution to a differential equation exists globally if it exists for all $t > 0$.

Which of the following statements is true?

(a.) Both $(*)$ and $(**)$ have global solutions

(b.) None of $(*)$ and $(**)$ have global solutions

(c.) There exists a global solution for $(*)$ and there exists a $T < \infty$ such that $\lim_{t \to T} |z(t)| = +\infty$

(d.) There exists a global solution for $(**)$ and there exists a $T<\infty$ such that $\lim_{t\to T}|y(t)|=+\infty$

(43.) The general solution of the surfaces which are perpendicular to the family of surfaces $z^2=kxy, k\in\mathbb{R}$ is

(a.) $\phi(x^2-y^2, xz)=0, \phi\in C^1(\mathbb{R}^2)$

(b.) $\phi(x^2-y^2, x^2+z^2)=0, \phi\in C^1(\mathbb{R}^2)$

(c.) $\phi(x^2-y^2, 2x^2+z^2)=0, \phi\in C^1(\mathbb{R}^2)$

(d.) $\phi(x^2+y^2, 3x^2-z^2)=0, \phi\in C^1(\mathbb{R}^2)$

(44.) The general solution of the equation $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$ is

(a.) $z=\phi\left(\frac{|x|}{|y|}\right), \phi\in C^1(\mathbb{R})$

(b.) $z=\phi\left(\frac{x-1}{y}\right), \phi\in C^1(\mathbb{R})$

(c.) $z=\phi\left(\frac{x+1}{y}\right), \phi\in C^1(\mathbb{R})$

(d.) $z=\phi(|x|+|y|), \phi\in C^1(\mathbb{R})$

(45.) Let f be an infinitely differentiable real-valued function on a bounded interval I. Take $n\ge 1$ interpolation points $\{x_0, x_1, \ldots, x_{n-1}\}$. Take n additional interpolation points $x_{n+j}=x_j+\varepsilon$, $j=0,1,\ldots,n-1$ where $\varepsilon>0$ is such that $\{x_0, x_1, \ldots, x_{2n-1}\}$ are all distinct.

Let p_{2n-1} be the Lagrange interpolation polynomial of degree $2n-1$ with the interpolation points $\{x_0, x_1, \ldots, x_{2n-1}\}$ for the function f.

Let q_{2n-1} be the Hermite interpolation polynomial of degree $2n-1$ with the interpolation points $\{x_0, x_1, \ldots, x_{n-1}\}$ for the function f. In the $\varepsilon\to 0$ limit the quantity $\sup_{x\in I}|p_{2n-1}(x)-q_{2n-1}(x)|$

(a.) Does not necessarily converge

(b.) Converges to $\frac{1}{2n}$

(c.) Converges to 0

(d.) Converges to $\frac{1}{2n+1}$

(46.) The extremal of the functional

$J(y)=\int_0^1\left[2(y')^2+xy\right]dx$,

$y(0)=0, y(1)=1, y\in C^2[0,1]$ is

(a.) $y=\frac{x^2}{12}+\frac{11x}{12}$

(b.) $y=\frac{x^3}{3}+\frac{2x^2}{3}$

(c.) $y=\frac{x^2}{7}+\frac{6x}{7}$

(d.) $y=\frac{x^3}{24}+\frac{23x}{24}$

(47.) The solution of the Fredholm integral equation $y(s)=s+2\int_0^1 (st^2+s^2t)\,y(t)\,dt$ is

(a.) $y(s)=-(50s+40s^2)$

(b.) $y(s)=(30s+15s^2)$

(c.) $y(s)=-(30s+40s^2)$

(d.) $y(s)=(60s+50s^2)$

(48.) Consider the solid S made of a material of constant density in the shape of a hemisphere of unit radius: $S=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\ \ z\ge 0\}$.

Which of the following statements is true?

(a.) The centre of mass of S is at the origin

(b.) The x-axis is a principal axis for S

(c.) The moment of inertia tensor of S is not a diagonal matrix

(d.) The z-axis is a principal axis for S

(49.) In an examination involving multiple choice questions, a student works out the solution in 50% of the questions. In the remaining questions the student guesses the answer. However, when the answer is guessed the probability that it is correct is 0.30. When the student works out the solutions it may be wrong with probability 0.10.

If the answer to a particular question is correct, what is the probability that the student guessed the answer?

(a.) 0.25

(b.) 0.50

(c.) 0.90

(d.) 0.30

(50.) Let $X_1, X_2, \ldots$ be i.i.d. random variables having a χ^2-distribution with 5 degrees of freedom.

Let $a\in\mathbb{R}$ be constant. Then the limiting distribution of $a\left(\frac{X_1+\ldots+X_n-5n}{\sqrt{n}}\right)$ is

(a.) Gamma distribution for an appropriate value of a

(b.) χ^2-distribution for an appropriate value of a

(c.) Standard normal distribution for an appropriate value of a

(d.) A degenerate distribution for an appropriate value of a

(51.) Consider a Markov Chain $X_0, X_1, X_2, \ldots$ with state space S. Suppose $i, j \in S$ are two states which communicate with each other. Which of the following statements is NOT correct?

(a.) Period of $i =$ period of j

(b.) i is recurrent if and only if j is recurrent

(c.) $\lim_{n\to\infty} P(X_n = i \mid X_0 = k) = \lim_{n\to\infty} P(X_n = j \mid X_0 = k)$ for all $k \in S$

(d.) $\lim_{n\to\infty} P(X_n = j \mid X_0 = i) = \lim_{n\to\infty} P(X_n = j \mid X_0 = j)$

(52.) Suppose that X has uniform distribution on the interval $[0, 100]$. Let Y denote the greatest integer smaller than or equal to X. Which of the following is true?

(a.) $P(Y \le 25) = 1/4$

(b.) $P(Y \le 25) = 26/100$

(c.) $E(Y) = 50$

(d.) $E(Y) = 101/2$

(53.) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common pdf $f(x \mid \theta) = \frac{(\log \theta)\theta^x}{\theta - 1}$, for $0 < x < 1$ where $\theta > 1$ is an unknown parameter. Then the statistic $T = \sum_{i=1}^{n} X_i$ is

(a.) Sufficient, but not complete

(b.) Sufficient, but not minimal sufficient

(c.) Complete sufficient

(d.) Neither complete, nor sufficient

(54.) Consider the pdf given by

$$f(x \mid \theta) = \frac{e^{(x-\theta)}}{[1 + e^{(x-\theta)}]^2},$$

$-\infty < x < \infty, -\infty < \theta < \infty$

Based on one observation X with the above pdf, a UMP test of size α for testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$ is

(a.) $X > k$ for some k such that $\alpha = P_{\theta_0}[X > k]$

(b.) $X < k$ for some k such that $\alpha = P_{\theta_0}[X > k]$

(c.) $X > k$ for some k such that $\alpha = P_{\theta_0}[X < k]$

(d.) $X < k$ for some k such that $\alpha = P_{\theta_0}[X < k]$

(55.) Consider 35 i.i.d. observations $X_1, X_2, \ldots, X_{15}$ and $Y_1, Y_2, \ldots, Y_{20}$. Let R be the Wilcoxon's rank sum statistic based on the ranks of the X's in the combined sample. Then the expected value of R is

(a.) 270

(b.) 300

(c.) 360.5

(d.) 330.5

(56.) Let $I, J > 5$. Consider two-way ANOVA, where the observations satisfy the linear model $y_{ij} = \alpha + \beta_i + \gamma_j + \varepsilon_{ij}$, $1 \le i \le I, 1 \le j \le J$

$E(\varepsilon_{ij}) = 0$, $Var(\varepsilon_{ij}) = \sigma^2$, $\sum_{i=1}^{I} \beta_i = \sum_{j=1}^{J} \gamma_j = 0$. In this set-up

(a.) β_1 is estimable

(b.) γ_1 is estimable

(c.) $\beta_1 - \beta_2$ is estimable

(d.) $\beta_1 + \gamma_2$ is estimable

(57.) Let X_1 and X_2 be two i.i.d. $p \times 1$ multivariate normal random vectors with mean μ and positive definite dispersion matrix Σ. Then which of the following random variables always has a central chi-square distribution

(a.) $\frac{1}{2}(X_1 - X_2)^T (X_1 - X_2)$

(b.) $2(X_1 - X_2)^T (X_1 - X_2)$

(c.) $2(X_1 - X_2)^T \Sigma^{-1} (X_1 - X_2)$

(d.) $\frac{1}{2}(X_1 - X_2)^T \Sigma^{-1} (X_1 - X_2)$

(58.) 10 units are chosen by simple random sampling without replacement from a population of size 100. Consider the sample variance $\frac{1}{10}\sum_{i=1}^{10}(y_i - \bar{y})^2 = s^2$. An unbiased estimate of population variance $\sigma^2 = \frac{1}{10}\sum_{i=1}^{100}(Y_i - \bar{Y})^2$ is

(a.) S^2

(b.) $\frac{10}{11}S^2$

(c.) $\frac{100}{99}S^2$

(d.) $\frac{100}{111}S^2$

(59.) Consider a Randomized Block Design with b blocks and k treatments. Let the observation corresponding to the i^{th} treatment and the j^{th} block be y_{ij}, $1 \le i \le k$, $1 \le j \le b$, which satisfies the usual linear model. Which of the following is true?

(a.) The estimates of any two treatment contrasts are uncorrelated

(b.) The error sum of squares has $bk-1$ degrees of freedom

(c.) The estimate of any treatment contrast is uncorrelated with the estimate of any block contrast

(d.) The correlation between the estimates of two treatment contrasts is always negative

(60.) The maximum and the minimum values of $5x+7y$, when $|x|+|y|\le 1$ are

(a.) 5 and –5

(b.) 5 and –7

(c.) 7 and –5

(d.) 7 and –7

PART-C

(Mathematical Sciences)

(61.) Which of the following sets are in bijection with $\mathbb{R}$?

(a.) Set of all maps from $\{0,1\}$ to $\mathbb{N}$

(b.) Set of all maps from $\mathbb{N}$ to $\{0,1\}$

(c.) Set of all subsets of $\mathbb{N}$

(d.) Set of all subsets of $\mathbb{R}$

(62.) Which of the following statements are true?

(a.) The series $\sum_{n\ge1}\frac{(-1)^n}{\sqrt{n}}$ is convergent

(b.) The series $\sum_{n\ge1}\frac{(-1)^n}{\sqrt{n}+n}$ is absolutely convergent

(c.) The series $\sum_{n\ge1}\frac{\left[1+(-1)^n\right]\sqrt{n}+\log n}{n^{3/2}}$ is convergent

(d.) The series $\sum_{n\ge1}\frac{\left((-1)^n\sqrt{n}+1\right)}{n^{3/2}}$ is convergent

(63.) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & (x,y)\ne(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Define $g(x,y)=\sum_{n=1}^{\infty}\frac{f\left((x-n),(y-n)\right)}{2^n}$.

Which of the following statements are true?

(a.) The function $h(y) = g(c, y)$ is continuous on $\mathbb{R}$ for all c

(b.) g is continuous from $\mathbb{R}^2$ into $\mathbb{R}$

(c.) g is not a well-defined function

(d.) g is continuous on $\mathbb{R}^2 \setminus \{(k,k)\}_{k\in\mathbb{N}}$

(64.) Consider two series $A(x) = \sum_{n=0}^{\infty} x^n (1-x)$ and $B(x) = \sum_{n=0}^{\infty} (-1)^n x^n (1-x)$ where $x \in [0,1]$.

Which of the following statements are true?

(a.) Both $A(x)$ and $B(x)$ converge pointwise

(b.) Both $A(x)$ and $B(x)$ converge uniformly

(c.) $A(x)$ converges uniformly but $B(x)$ does not

(d.) $B(x)$ converges uniformly but $A(x)$ does not

(65.) For $p \in \mathbb{R}$, consider the improper integral $I_p = \int_0^1 t^p \sin t \, dt$. Which of the following statements are true?

(a.) I_p is convergent for $p = -1/2$

(b.) I_p is divergent for $p = -3/2$

(c.) I_p is convergent for $p = 4/3$

(d.) I_p is divergent for $p = -4/3$

(66.) Suppose that $\{f_n\}$ is a sequence of real-valued functions on $\mathbb{R}$. Suppose it converges to a continuous function f uniformly on each closed and bounded subset of $\mathbb{R}$. Which of the following statements are true?

(a.) The sequence $\{f_n\}$ converges to f uniformly on $\mathbb{R}$

(b.) The sequence $\{f_n\}$ converges to f pointwise on $\mathbb{R}$

(c.) For all sufficiently large n, the function f_n is bounded

(d.) For all sufficiently large n the function f_n is continuous

(67.) Let $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$.

Which of the following statements are true?

(a.) Both f and g are uniformly continuous on $\mathbb{R}$

(b.) f is uniformly continuous on every interval of the form $[a, +\infty), a \in \mathbb{R}$

(c.) g is uniformly continuous on $\mathbb{R}$

(d.) $f(x)g(x)$ is uniformly continuous on $\mathbb{R}$

(68.) Define $f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$

Which of the following statements are true?

(a.) f is discontinuous at $(0,0)$

(b.) f is continuous at $(0,0)$

(c.) All directional derivatives of f at $(0,0)$ exist

(d.) f is not differentiable at $(0,0)$

(69.) Define $f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$

Which of the following statements are true?

(a.) f is continuous at $(0,0)$

(b.) f is bounded in a neighbourhood of $(0,0)$

(c.) f is not bounded in any neighbourhood of $(0,0)$

(d.) f has all directional derivatives at $(0,0)$

(70.) Let $p:\mathbb{R}^2\to\mathbb{R}$ be defined by

$p(x,y)=\begin{cases}|x| & \text{if } x\neq 0\\ |y| & \text{if } x=0\end{cases}.$

Which of the following statements are true?

(a.) $p(x,y)=0$ if and only if $x=y=0$

(b.) $p(x,y)\geq 0$ for all x,y

(c.) $p(\alpha x,\alpha y)=|\alpha|\,p(x,y)$ for all $\alpha\in\mathbb{R}$ and for all x,y

(d.) $p(x_1+x_2,y_1+y_2)\leq p(x_1,y_1)+p(x_2,y_2)$ for all $(x_1,y_1),(x_2,y_2)$

(71.) Let P be a square matrix such that $P^2=P$. Which of the following statements are true?

(a.) Trace of P is an irrational number

(b.) Trace of $P=$ rank of P

(c.) Trace of P is an integer

(d.) Trace of P is an imaginary complex number

(72.) Let A and B be $n\times n$ real matrices and let $C=\begin{pmatrix}A & B\\ B & A\end{pmatrix}$. Which of the following statements are true?

(a.) If λ is an eigenvalue of $A+B$ then λ is an eigenvalue of C

(b.) If λ is an eigenvalue of $A-B$ then λ is an eigenvalue of C

(c.) If λ is an eigenvalue of A or B then λ is an eigenvalue of C

(d.) All eigenvalues of C are real

(73.) Let A be an $n\times n$ real matrix. Let b be an $n\times 1$ vector. Suppose $Ax=b$ has no solution. Which of the following statements are true?

(a.) There exists an $n\times 1$ vector c such that $Ax=c$ has a unique solution

(b.) There exist infinitely many vectors c such that $Ax=c$ has no solution

(c.) If y is the first column of A then $Ax=y$ has a unique solution

(d.) $\det A=0$

(74.) Let A be an $n\times n$ matrix such that the first 3 rows of A are linearly independent and the first 5 columns of A are linearly independent. Which of the following statements are true?

(a.) A has at least 5 linearly independent rows

(b.) $3\le \text{rank } A\le 5$

(c.) Rank $A\ge 5$

(d.) Rank $A^2\ge 5$

(75.) Let n be a positive integer and F be a non-empty proper subset of $\{1,2,\ldots,n\}$. Define $\langle x,y\rangle_F=\sum_{k\in F}x_k y_k$, $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$. Let $T=\{x\in\mathbb{R}^n:\langle x,x\rangle_F=0\}$.

For $y\in\mathbb{R}^n, y\neq 0$

Which of the following statements are true?

(a.) $\inf_{x\in T}\langle x+y,x+y\rangle_F=\langle y,y\rangle_F$

(b.) $\sup_{x\in T}\langle x+y,x+y\rangle_F=\langle y,y\rangle_F$

(c.) $\inf_{x\in T}\langle x+y,x+y\rangle_F<\langle y,y\rangle_F$

(d.) $\sup_{x\in T}\langle x+y,x+y\rangle_F>\langle y,y\rangle_F$

(76.) Let $v\in\mathbb{R}^3$ be a non-zero vector. Define a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ by $T(x)=x-2\dfrac{x\cdot v}{v\cdot v}v$, where $x\cdot y$ denotes the standard inner product in $\mathbb{R}^3$.

Which of the following statements are true?

(a.) The eigenvalues of T are $+1,-1$

(b.) The determinant of T is -1

(c.) The trace of T is $+1$

(d.) T is distance preserving

(77.) A quadratic form $Q(x,y,z)$ over $\mathbb{R}$ represents 0 non trivially if there exists $(a,b,c)\in\mathbb{R}^3\setminus\{(0,0,0)\}$ such that $Q(a,b,c)=0$. Which of the following quadratic forms $Q(x,y,z)$ over $\mathbb{R}$ represent 0 non trivially?

(a.) $Q(x,y,z)=xy+z^2$

(b.) $Q(x,y,z)=x^2+3y^2-2z^2$

(c.) $Q(x,y,z)=x^2-xy+y^2+z^2$

(d.) $Q(x,y,z)=x^2+xy+z^2$

(78.) Let $Q(x,y,z)$ be a real quadratic form. Which of the following statements are true?

(a.) $Q(x_1+x_2,y,z)=Q(x_1,y,z)+Q(x_2,y,z)$ for all x_1,x_2,y,z

(b.) $Q(x_1+x_2,y_1+y_2,0)+Q(x_1-x_2,y_1-y_2,0)=2Q(x_1,y_1,0)+2Q(x_2,y_2,0)$ for all x_1,x_2,y_1,y_2

(c.) $Q(x_1+x_2,y_1+y_2,z_1+z_2)=Q(x_1,y_1,z_1)+Q(x_2,y_2,z_2)$ for at least one choice of x_1,x_2,y_1,y_2,z_1,z_2

(d.) $2Q(x_1+x_2,y_1+y_2,0)+2Q(x_1-x_2,y_1-y_2,0)=Q(x_1,y_1,0)+Q(x_2,y_2,0)$ for all x_1,x_2,y_1,y_2

(79.) For $z\neq -i$, let $f(x)=\exp\left(\frac{1}{z+i}\right)-1$.

Which of the following are true?

(a.) f has finitely many zeros

(b.) f has a sequence of zeros that converges to a removable singularity of f

(c.) f has a sequence of zeros that converges to a pole of f

(d.) f has a sequence of zeros that converges to an essential singularity of f

(80.) Let f be a holomorphic function on the open unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. Suppose that $|f|\geq 1$ on $\mathbb{D}$ and $f(0)=i$. Which of the following are possible values of $f\left(\frac{1}{2}\right)$?

(a.) $-i$

(b.) i

(c.) 1

(d.) -1

(81.) Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disc and let $f:\mathbb{D}\to\mathbb{D}$ be a holomorphic function. Suppose that $f(0)=0$ and $f'(0)=0$. Which of the following are possible values of $f\left(\frac{1}{2}\right)$?

(a.) $\frac{1}{4}$

(b.) $-\frac{1}{4}$

(c.) $\frac{1}{3}$

(d.) $-\frac{1}{3}$

(82.) Let n be a positive integer. For a real number $R > 1$, let $z(\theta) = R\, e^{i\theta}$, $0 \le \theta < 2\pi$. The set $\{\theta \in [0, 2\pi) : |z(\theta)^n + 1| = |z(\theta)^n| - 1\}$ contains which of the following sets?

(a.) $\{\theta \in [0, 2\pi) : \cos n\theta = 1\}$

(b.) $\{\theta \in [0, 2\pi) : \sin n\theta = 1\}$

(c.) $\{\theta \in [0, 2\pi) : \cos n\theta = -1\}$

(d.) $\{\theta \in [0, 2\pi) : \sin n\theta = -1\}$

(83.) Which of the following statements are true?

(a.) $\mathbb{Q}$ has countably many subgroups

(b.) $\mathbb{Q}$ has uncountably many subsets

(c.) Every finitely generated subgroup of $\mathbb{Q}$ is cyclic

(d.) $\mathbb{Q}$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}$ as groups

(84.) Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}$

and for any prime p, let

$$\Gamma(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{array}{l} a \equiv 1 (\operatorname{mod} p), d \equiv 1 (\operatorname{mod} p) \\ c \equiv 0 (\operatorname{mod} p), b \equiv 0 (\operatorname{mod} p) \end{array} \right\}.$$

Which of the following are true?

(a.) $\Gamma(p)$ is a subgroup of $SL_2(\mathbb{Z})$

(b.) $\Gamma(p)$ is not a normal subgroup of $SL_2(\mathbb{Z})$

(c.) $\Gamma(p)$ has atleast two elements

(d.) $\Gamma(p)$ is uncountable

(85.) Let G be a finite group.

Which of the following are true?

(a.) If $g \in G$ has order m and if $n \ge 1$ divides m, then G has a subgroup of order n

(b.) If for any two subgroups A and B of G, either $A \subset B$ or $B \subset A$, then G is cyclic

(c.) If G is cyclic, then for any two subgroups A and B of G, either $A \subset B$ or $B \subset A$

(d.) If for every positive integer m dividing $|G|$, G has a subgroup of order m, then G is abelian

(86.) Let R,S be commutative rings with unity, $f:R\to S$ be a surjective ring homomorphism, $Q\subseteq S$ be a non-zero prime ideal. Which of the following statements are true?

(a.) $f^{-1}(Q)$ is a non-zero prime ideal in R

(b.) $f^{-1}(Q)$ is a maximal ideal in R if R is a PID

(c.) $f^{-1}(Q)$ is a maximal ideal in R if R is a finite commutative ring with unity

(d.) $f^{-1}(Q)$ is a maximal ideal in R if $x^5=x$ for all $x\in R$

(87.) Consider the polynomial $f(x)=x^2+3x-1$. Which of the following statements are true?

(a.) f is irreducible over $\mathbb{Z}[\sqrt{13}]$

(b.) f is irreducible over $\mathbb{Q}$

(c.) f is reducible over $\mathbb{Q}[\sqrt{13}]$

(d.) $\mathbb{Z}[\sqrt{13}]$ is a unique factorization domain

(88.) Let p be an odd prime such that $p\equiv 2(\bmod 3)$. Let $\mathbb{F}_p$ be the field with p elements. Consider the subset E of $\mathbb{F}_p\times\mathbb{F}_p$ given by $E=\{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p : y^2=x^3+1\}$.

Which of the following are true?

(a.) E has atleast two elements

(b.) E has atmost $2p$ elements

(c.) E can have p^2 elements

(d.) E has atleast $2p$ elements

(89.) Consider the subset of $\mathbb{R}^2$ defined as follows: $A=\{(x,y)\in\mathbb{R}\times\mathbb{R}:(x-1)(x-2)(y-3)(y+4)=0\}$. Which of the following statements are true?

(a.) A is connected

(b.) A is compact

(c.) A is closed

(d.) A is dense

(90.) Let X be a non-empty set. Suppose that τ_1 and τ_2 are two topologies over X, such that $\tau_2\subset\tau_1$. Which of the following statements imply $\tau_1=\tau_2$?

(a.) (X,τ_1) is compact and τ_1 is T_2 (Hausdorff)

(b.) (X,τ_1) is compact and τ_2 is T_2 (Hausdorff)

(c.) The connected components of both (X,τ_1) and (X,τ_2) are same

(d.) For any subset $A\subset X$ the closure of A in (X,τ_2) is contained in the closure of A in (X,τ_1)

(91.) The following two-point boundary value problem

$$\begin{cases} y''(x)+\lambda y(x)=0 \text{ for } x\in(0,\pi) \\ y(0)=0 \\ y(\pi)=0 \end{cases}$$

has a trivial solution $y=0$. It also has a non-trivial solution for

(a.) No values of $\lambda\in\mathbb{R}$

(b.) $\lambda=1$

(c.) $\lambda=n^2$ for all $n\in\mathbb{N}, n>1$

(d.) $\lambda\le 0$

(92.) Let A be an $n\times n$ matrix with distinct eigenvalues $(\lambda_1,\ldots,\lambda_n)$ with corresponding linearly independent eigen vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$. Then, the non-homogeneous differential equation $\mathbf{x}'(t)=A\mathbf{x}(t)+e^{\lambda_1 t}\mathbf{v}_1$

(a.) Does not have a solution of the form $e^{\lambda_1 t}\mathbf{a}$ for any vector $\mathbf{a}\in\mathbb{R}^n$

(b.) Has a solution of the form $e^{\lambda_1 t}\mathbf{a}$ for some vector $\mathbf{a}\in\mathbb{R}^n$

(c.) Has a solution of the form $e^{\lambda_1 t}\mathbf{a}+te^{\lambda_1 t}\mathbf{b}$ for some vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$

(d.) Does not have a solution of the form $e^{\lambda_1 t}\mathbf{a}+te^{\lambda_1 t}\mathbf{b}$ for any vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$

(93.) Consider the solutions $y_1:=\begin{pmatrix} e^{-3t} \\ e^{-3t} \\ 0 \end{pmatrix}$ and $y_2:=\begin{pmatrix} 0 \\ e^{-5t} \\ e^{-5t} \end{pmatrix}$ to the homogeneous linear system of differential equation

$$(*)\ y'(t)=\begin{pmatrix} -5 & 2 & -2 \\ 1 & -4 & -1 \\ -1 & 1 & -6 \end{pmatrix} y(t).$$

Which of the following statements are true?

(a.) y_1 and y_2 form a basis for the set of all solutions to $(*)$

(b.) y_1 and y_2 are linearly independent but do not form a basis for the set of all solutions to $(*)$

(c.) There exists another solution y_3 such that $\{y_1,y_2,y_3\}$ form a basis for the set of all solutions to $(*)$

(d.) y_1 and y_2 are linearly dependent

(94.) Consider the partial differential equations

(i) $\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}+(1-\operatorname{sgn}(y))\dfrac{\partial^2 u}{\partial y^2}=0$

(ii) $y\dfrac{\partial^2 u}{\partial x^2}+x\dfrac{\partial^2 u}{\partial y^2}=0$

Which of the following statements are true?

(a.) Equation (i) is parabolic for $y>0$ and elliptic for $y<0$

(b.) Equation (i) is hyperbolic for $y>0$ and elliptic for $y<0$

(c.) Equation (ii) is elliptic in I and III quadrant and hyperbolic in II and IV quadrant

(d.) Equation (ii) is hyperbolic in I and III quadrant and elliptic in II and IV quadrant

(95.) Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 0, |x|<1, 0<y<1 \\ u(x,x^2)=0, \quad \dfrac{\partial u}{\partial y}(x,x^2)=g(x), |x|<1 \end{cases}$$

Which of the following statements are true?

(a.) A necessary condition for a solution to exist is that g is an odd function

(b.) A necessary condition for a solution to exist is that g is an even function

(c.) The solution (if it exists) is given by $u(x,y)=2\int_{x}^{\sqrt{y}} zg(z)\,dz$

(d.) The solution (if it exists) is given by $u(x,y)=2\int_{\sqrt{y}}^{x^2} zg(z)dz$

(96.) Fix a $\alpha \in (0,1)$. Consider the iteration defined by $(*)$ $x_{k+1}=\frac{1}{2}(x_k^2+\alpha)$, $k=0,1,2,\ldots$

The above iteration has two distinct fixed points ζ_1 and ζ_2 such that $0<\zeta_1<1<\zeta_2$. Which of the following statements are true?

(a.) The iteration $(*)$ is equivalent to the recurrence relation $x_{k+1}-\zeta_1 = \frac{1}{2}(x_k+\zeta_1)\ (x_k-\zeta_1)$, $k=0,1,2,\ldots$

(b.) The iteration $(*)$ is equivalent to the recurrence relation $x_{k+1}-\zeta_1 = \frac{1}{2}(x_k+\zeta_2)\ (x_k-\zeta_1)$, $k=0,1,2,\ldots$

(c.) If $0\le x_0<\zeta_2$ then $\lim_{k\to\infty} x_k=\zeta_1$

(d.) If $-\zeta_2<x_0\le 0$ then $\lim_{k\to\infty} x_k=\zeta_1$

(97.) Consider the function $f:[0,1]\to\mathbb{R}$ defined by $f(x):=\begin{cases} 2^{-\left\{1+\left(\log_2\left(\frac{1}{x}\right)\right)^{1/\beta}\right\}^{\beta}} & \text{for } x\in(0,1] \\ 0 & \text{for } x=0, \end{cases}$

where $\beta\in(0,\infty)$ is a parameter. Consider the iterations $x_{k+1}=f(x_k)$, $k=0,1,\ldots;x_0>0$. Which of the following statements are true about the iteration?

(a.) For $\beta = 1$, the sequence $\{x_k\}$ converges to 0 linearly with asymptotic rate of convergence $\log_{10} 2$

(b.) For $\beta > 1$, the sequence $\{x_k\}$ does not converge to 0

(c.) For $\beta \in (0,1)$, the sequence $\{x_k\}$ converges to 0 sub-linearly

(d.) For $\beta \in (0,1)$, the sequence $\{x_k\}$ converges to 0 super-linearly

(98.) The extremal of the functional

$$J(y) = \int_0^1 e^x \sqrt{1+(y')^2}\, dx,\ y \in C^2[0,1]$$

is of the form

(a.) $y = \sec^{-1}\left(\frac{x}{c_1}\right) + c_2$, where c_1 and c_2 are arbitrary constants

(b.) $y = \sec^{-1}\left(\frac{x}{c_1}\right) + c_2$, where $|c_1| < 1$ and c_2 is an arbitrary constant

(c.) $y = \tan^{-1}\left(\frac{x}{c_1}\right) + c_2$, where c_1 and c_2 are arbitrary constants

(d.) $y = \tan^{-1}\left(\frac{x}{c_1}\right) + c_2$, where $|c_1| > 1$ and c_2 are arbitrary constants

(99.) Consider the functional

$$J(y) = \int_0^\pi \left((y')^2 - ky^2\right) dx \text{ with boundary conditions } y(0) = 0, y(\pi) = 0$$

Which of the following statements are true?

(a.) It has a unique extremal for all $k \in \mathbb{R}$

(b.) It has at most one extremal if $\sqrt{k}$ is not an integer

(c.) It has infinitely many extremals if $\sqrt{k}$ is an integer

(d.) It has a unique extremal if $\sqrt{k}$ is an integer

(100.) For the Fredholm integral equation

$$y(s) = \lambda \int_0^1 e^s e^t y(t)\, dt$$

Which of the following statements are true?

(a.) It has a non-trivial solution satisfying $\int_0^1 e^t y(t)\, dt = 0$

(b.) Only the trivial solution satisfies $\int_0^1 e^t y(t)\, dt = 0$

(c.) It has non-trivial solution for all $\lambda \neq 0$

(d.) It has non-trivial solutions only if $\lambda = \frac{2}{e^2-1}$ and $\int_0^1 e^t y(t)\,dt \neq 0$

(101.) Consider the partial differential equation

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

Which of the following statements are true?

(a.) The complete integral is $z = xa + yb + ab$, a, b arbitrary constants

(b.) The complete integral is $z = xa + yb + \sqrt{a^2+b^2}$, a, b arbitrary constants

(c.) The particular solution passing through $x = 0$ and $z = y^2$ is $\left(\frac{x}{4} - y\right)^2$

(d.) The particular solution passing through $x = 0$ and $z = y^2$ is $\left(\frac{x}{4} + y\right)^2$

(102.) Consider a dynamical system with the Lagrangian function $L(q,\dot{q}) = T - U$, where the kinetic energy $T = a(q)\dot{q}^2 \geq 0$ and the potential energy $U := U(q)$ and $a(q) > 0$. Which of the following statements are true?

(a.) The associated Lagrange's equation is $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

(b.) The associated Lagrange's equation is $\frac{d}{dt}\frac{\partial L}{\partial q} = \frac{\partial L}{\partial \dot{q}}$

(c.) The point $(q_0, \dot{q}_0)$ is an equilibrium position of the dynamical system if and only if $\dot{q}_0 = 0$ and $\left.\frac{\partial U}{\partial q}\right|_{q=q_0} = 0$

(d.) The point $(q_0, \dot{q}_0)$ is an equilibrium position of the dynamical system if and only if $\dot{q}_0 = 0$ and $\left.\frac{\partial U}{\partial q}\right|_{q=q_0} > 0$

(103.) Let X and Y be independent random variables with $E(X) = E(Y) = 0$ and $Var(X) = Var(Y) = 1$. Let $Z = X + Y$. Which of the following statements are correct?

(a.) $P(|Z| > \varepsilon) \leq 2/\varepsilon^2$

(b.) $P(|Z|) \leq \sqrt{2}$

(c.) $P(Z) = 2$

(d.) $P(Z \leq 0) = P(Z \geq 0)$

(104.) Let $n > 1$, let $X_1, X_2, .., X_n$ be random variables such that $E(X_i) = 0$ and $E(X_i^2) = 1$ for all i and $E(X_i X_j) = \rho$ for all $i \neq j$. Which of the following statements are true?

(a.) $\rho = 0$ if and only if $X_1, X_2, ...X_n$ are independent

(b.) $Var(X_1 + X_2 + ... + X_n) = n$ if and only if $X_1, X_2, .., X_n$ are independent

(c.) $Var(X_1 + X_2 + ... + X_n) = n$ if and only if $X_1, X_2, .., X_n$ are pairwise

(d.) $Var(X_1 + X_2 + ... + X_n) = n$ if and only if $\rho = 0$

(105.) Consider a Markov chain with a countable state space S. Identify the correct statements.

(a.) If the Markov chain is aperiodic and irreducible then there exists a stationary distribution

(b.) If the Markov chain is aperiodic and irreducible then there is at most one stationary distribution

(c.) If S is finite then there exists a stationary distribution

(d.) If S is finite then there is exactly one stationary distribution

(106.) Consider a Markov chain with transition probability matrix

$$P = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \end{pmatrix}$$

Let $\pi = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$.

Then which of the following statements are correct?

(a.) π is a stationary distribution

(b.) If η is a stationary distribution, then $\eta = \pi$

(c.) The Markov chain is periodic

(d.) The Markov chain is irreducible

(107.) Suppose $X_1, X_2, ..., X_n$ are i.i.d. random variables with characteristic function $\phi(t;\theta) = E[e^{itX_1}]$ where $\underline{\theta} \in \mathbb{R}^k$ is the parameter of the distribution. Let $Z = X_1 + X_2 + ... + X_n$. Then for which of the following distributions of X_1 would the characteristic function of Z be of the form $\phi(t;\underline{a})$ for some $\underline{\alpha} \in \mathbb{R}^k$?

(a.) Negative Binomial

(b.) Geometric

(c.) Hypergeometric

(d.) Discrete Uniform

(108.) Let $X_1, X_2, ..., X_n$ be i.i.d. with the common pdf $f(x|\theta) = \frac{\theta}{x^{\theta+1}}$, for $x > 1$ where $\theta > 1$ is an unknown parameter. Which of the following estimators of θ are consistent?

(a.) $\frac{1}{n}\sum_{i=1}^{n} X_i$

(b.) $\frac{1}{n}\sum_{i=1}^{n}\log(X_i)$

(c.) $\frac{n}{\sum_{i=1}^{n}X_i}$

(d.) $\frac{n}{\sum_{i=1}^{n}\log(X_i)}$

(109.) Let $X_1, X_2, \ldots, X_n$ be i.i.d. with the common probability mass function $p(x\,|\,\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0$ or 1, and $0 \le \theta \le \frac{1}{2}$. Then

(a.) The method of moments estimator of θ is $\frac{1}{2n}\sum_{i=1}^{n}X_i$

(b.) The MLE of θ is $\min_{1\le i\le n} X_i$

(c.) The method of moments estimator of θ is $\min_{1\le i\le n} X_i$

(d.) The MLE of θ is $\min\left\{\frac{1}{n}\sum_{i=1}^{n}X_i, \frac{1}{2}\right\}$

(110.) Let the pdf of X by $f(x\,|\,\theta) = \frac{2x}{\theta^2}$, for $0 < x < \theta$, where $\theta > 0$ is unknown parameter. Which of the following are $100(1-\alpha)\%$ confidence intervals for θ?

(a.) $\left[X, \frac{X}{\sqrt{\alpha}}\right]$

(b.) $[X, 2X]$

(c.) $\left[\frac{\sqrt{2}}{\sqrt{2-\alpha}}X, \frac{\sqrt{2}}{\sqrt{\alpha}}X\right]$

(d.) $[0, X]$

(111.) X has binomial distribution with parameters n and p. Suppose that n is given and the unknown parameter p has prior distribution, which is uniform on the interval $[0,1]$. Consider the squared error loss function and the observation $X = n$. Identify the correct statement.

(a.) The Bayes estimate of p is $\left(\frac{n+1}{n+2}\right)$

(b.) The Bayes estimate of p is $2^{-1/(n+1)}$

(c.) The median of the posterior distribution of p is $2^{-1/(n+1)}$

(d.) The median of the posterior distribution of p is $\left(\frac{n+1}{n+2}\right)$

(112.) Let X_1, X_2, X_3 be a random sample from the uniform distribution on the interval $(0,\theta)$. Suppose the prior distribution of θ is uniform on the interval $(0,1)$. Let $X_{(3)} = \max\{X_1, X_2, X_3\}$. Consider the squared error loss function. Which of the following statements are necessarily true?

(a.) Bayes estimator of θ is unique

(b.) $\frac{1}{X_{(3)}}$ is a Bayes estimator of θ

(c.) $X_{(3)}$ is a Bayes estimator of θ

(d.) $\frac{1-X_{(3)}}{X_{(3)}}$ is a Bayes estimator of θ

(113.) Consider the Gauss-Markov model $\mathbf{Y}_{n\times 1} = \mathbf{X}_{n\times p}\boldsymbol{\beta}_{p\times 1} + \boldsymbol{\varepsilon}_{n\times 1}$, where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and Dispersion $(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}_{n\times n}$. Suppose that $p < n$. Which of the following are correct?

(a.) Least-squares estimate of $\boldsymbol{\beta}$ is unique

(b.) Least-squares estimate of an estimable linear function of $\boldsymbol{\beta}$ is unique

(c.) Least-squares estimate of $\mathbf{X}\boldsymbol{\beta}$ is unique

(d.) Determinant $(\mathbf{X}^T\mathbf{X}) > 0$

(114.) Let $p > 1$ and $1 > \rho \geq 0$. Consider a multiple linear regression problem with p independent variables $X_1, X_2, \ldots, X_p$ and a dependent variable Y. Suppose that the correlation between Y and X_i is ρ and the correlation between X_i and X_j is also ρ for all $1 \leq i < j \leq p$. Which of the following are correct?

(a.) The multiple correlation between Y and $(X_1, \ldots, X_p)$ is larger than or equal to ρ

(b.) The multiple correlation between Y and $(X_1, \ldots, X_p)$ will be ρ if $\rho = 0$

(c.) The multiple correlation between Y and $(X_1, \ldots, X_p)$ will be ρ only if $\rho = 0$

(d.) The multiple correlation between Y and $(X_1, \ldots, X_p)$ tends to 1 as $p \to \infty$

(115.) Let $n > 2$ and $0 < \theta < \frac{\pi}{2}$ be fixed. Let $X_1, \ldots, X_n$ be i.i.d. normal random variables with mean zero and variance $\sigma^2 > 0$. For $i = 1, \ldots, n$ define $Y_{2i-1} = X_i \cos\theta$ and $Y_{2i} = X_i \sin\theta$. Further, let $Z^T = (Y_1, Y_2, \ldots, Y_{2n})$, and $V^T = (X_1, Y_1, Y_2, X_2, Y_3, Y_4, \ldots, X_n, Y_{2n-1}, Y_{2n})$. Which of the following statements are correct?

(a.) Z^T has a multivariate normal distribution

(b.) There exists a constant C, such that CZ^TZ has a chi-square distribution

(c.) V^T has a multivariate normal distribution

(d.) $E\left(\frac{1}{V^TV}\right) < \infty$

(116.) For circular systematic sampling, which of the following are correct?

(a.) Sample mean is an unbiased estimate for population mean but sample variance is not an unbiased estimate for population variance (b.) Sample mean and sample variance are unbiased estimates for population mean and population variance respectively

(c.) Sample mean is not an unbiased estimate for population mean but sample variance is an unbiased estimate for population variance

(d.) Neither sample mean nor sample variance is an unbiased estimate for their population counterparts

(117.) In a Randomized Block Design with one observation per cell, and data satisfying the standard linear model, which of the following are correct?

(a.) Mean treatment effects are estimable

(b.) Mean block effects are estimable

(c.) Treatment-Block interactions are NOT estimable

(d.) Treatment and block effects as well as treatment-block interactions are estimable

(118.) Suppose $\lambda(t)$ for $t \geq 0$ is a continuous hazard function of a non-negative random variable X, where $\lambda(t) \geq 1$. Which of the following statements are always true?

(a.) $\frac{1}{\lambda(t)}$ is a hazard function

(b.) $\lambda^2(t)$ is also a hazard function

(c.) $c\lambda(t)$ for $c \geq 0$ is also a hazard function

(d.) $\log \lambda(t)$ is a hazard function

(119.) Let $X_i = \theta + \varepsilon_i, 1 \leq i \leq n$ where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are i.i.d., with pdf $g(\varepsilon) = |\varepsilon|$, $-1 < \varepsilon < 1$. Let $T_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T_2 = X_{\left(\left[\frac{3n}{4}\right]+1\right)}$, the sample 75th percentile. Which of the following are correct?

(a.) T_1 is a consistent and asymptotically normal estimator of θ

(b.) $T_2 - \frac{1}{\sqrt{2}}$ is a consistent and asymptotically normal estimator of θ

(c.) The asymptotic variance of T_1 is $\frac{1}{2n}$

(d.) The asymptotic variance of T_2 is $\frac{3}{8n}$

(120.) Consider a M/M/1 queue with arrival rate λ and service rate μ. Let $Q_0 = 0$ and Q_t denote the queue length at time t. Which of the following statements are true?

(a.) (Q_t) admits a stationary distribution if and only if $\lambda \leq \mu$

(b.) The stationary distribution of the process (Q_t) is geometric, when it exists

(c.) $\lim_{t\to\infty} P(Q_t > k) = 1$ for all $k < \infty$ if $\lambda > \mu$

(d.) $\lim_{t\to\infty} P(Q_t > k) = 2^{-(k+1)}$ for all $k < \infty$ if $\lambda = \frac{\mu}{2}$